AN APPROXIMATE METHOD OF SOLVING PROBLEMS OF MAGNETOELASTICITY FOR BODIES OF FINITE CONDUCTIVITY

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1. This note is concerned with the question of magnetoelastic vibration of bodies of finite, but small conductivity in sufficiently strong magnetic fields. The induced magnetic field is neglected in comparison with the applied uniform field. However, the ponderomotive forces are assumed to be quantities of the same order as the body and inertia forces (this approach is analogous to the case of small magnetic Reynolds numbers, but finite interaction parameters in magnetohydrodynamics). With the above assumptions the electric field may be considered potential

$$\mathbf{E} = -\operatorname{grad} f. \tag{1}$$

The relation between the potential $f$ and the displacement $\mathbf{u}$ of points in the medium is given by

$$\Delta f + \frac{\varepsilon}{4\pi\sigma}\frac{\partial \Delta f}{\partial t} = \frac{\mu}{c}\operatorname{div}\frac{\partial \mathbf{u}}{\partial t}\times\mathbf{H}. \tag{2}$$

This relation is obtained by the divergence operation from the equation

$$4\pi\sigma\left(\mathbf{E} + \frac{\mu}{c}\frac{\partial u}{\partial t}\times\mathbf{H}\right) = c\operatorname{rot}\mathbf{H} - \varepsilon\frac{\partial \mathbf{E}}{\partial t}. \tag{3}$$

Here $\mathbf{H}$ is the external magnetic field, $\varepsilon$ is the dielectric constant, $\mu$ is the magnetic permeability, $\sigma$ the conductivity, c the speed of light. The dynamic equations of the theory of elasticity have the form

$$G\Delta\mathbf{u} + (\lambda + G)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{F} + \mathbf{P} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{4}$$

Here $\lambda$ and G are the Lamé constants, F is the body force, $\rho$ the density; the ponderomotive force P is given by the expression

$$P = \frac{\mu\sigma}{c}\left\{\frac{\mu}{c}\left[\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}\right]\times\mathbf{H} + \mathbf{H}\times\operatorname{grad} f\right\}. \tag{5}$$

Thus, in the approximation considered the problem reduces to the determination of the displacement u and the potential $f$ from equations (2) and (4), after which the induced magnetic field $\mathbf{h}$ must be found from the relation

$$\Delta\mathbf{h} = -\frac{4\pi\sigma\mu}{c^2}\operatorname{rot}\left[\frac{\partial\mathbf{u}}{\partial t}\times\mathbf{H}\right]. \tag{6}$$

2. We shall apply the proposed method to the special case of plane vibrations of an infinite elastic body in a transverse magnetic field, when $\partial/\partial z = 0$, $u_z = 0$, $\mathbf{H} = H\mathbf{k}$. In this case $\mathbf{h} = \mathbf{k}h$, $\mathbf{H}\times\operatorname{grad} f = -H\operatorname{rot} f\mathbf{k}$, $[\mathbf{u}\times\mathbf{H}]\times\mathbf{H} = -H^2\mathbf{u}$, and expression (5) assumes the form

$$\mathbf{P} = -\frac{\mu\sigma H}{c}\left(\frac{\mu H}{c}\frac{\partial\mathbf{u}}{\partial t} + \operatorname{rot} f\mathbf{k}\right). \tag{7}$$

We introduce the elastic potentials $\varphi$ and $\psi$ from the usual formulas $\mathbf{u} = \operatorname{grad}\varphi + \operatorname{rot}\psi\mathbf{k}$ and put $F = \operatorname{grad}\Phi + \operatorname{rot}\Psi\mathbf{k}$. The basic equation (4) breaks down into the two components

$$(\lambda + G)\Delta\varphi - \frac{\sigma\mu^2 H^2}{c^2}\frac{\partial\varphi}{\partial t} + \Phi = \rho\frac{\partial^2\varphi}{\partial t^2}, \tag{8}$$

$$G\Delta\psi - \frac{\mu\sigma H}{c}\left(\frac{\mu H}{c}\frac{\partial\psi}{\partial t} + f\right) + \Psi = \rho\frac{\partial^2\psi}{\partial t^2}. \tag{9}$$

On the basis of the equality $\operatorname{div}\mathbf{u}\times\mathbf{H} = H\operatorname{rot}\mathbf{u}$ and the condition that the potentials vanish at infinity, we may write relation (2) in the form

$$f + \frac{\varepsilon}{4\pi\sigma}\frac{\partial f}{\partial t} + \frac{\mu H}{c}\frac{\partial\psi}{\partial t} = 0. \tag{10}$$

Therefore Eq. (9) can be represented in a somewhat different form

$$G\Delta\psi + \frac{\mu\varepsilon H}{4\pi c}\frac{\partial f}{\partial t} + \Psi = \rho\frac{\partial^2\psi}{\partial t^2}. \tag{11}$$

Thus, the potential $\varphi$ of the expansion waves is given by Eq. (8), while the potentials $\psi$ and j of the shear waves and the electric field are given by the system of equations (10), (11). After these have been found, we get the induced magnetic field from relation (6) and the equality rot [$\mathbf{u} \times \mathbf{H}$] = $-$H div $\mathbf{u}$

$$h = \frac{4\pi\sigma\mu H}{c^2}\frac{\partial\varphi}{\partial t}. \tag{12}$$

The results obtained show, first of all, that in the case in question the expansion waves and the induced magnetic field do not depend on the displacement current, since Eq. (8) does not contain the parameter $\varepsilon$. The effect of the conductivity of the medium is manifested in the form of the dissipative term in Eq. (8), which is proportional to the derivative with respect to time of the expansion wave potential. As far as the shear waves are concerned, they are related with both the parameters $\sigma$ and $\varepsilon$. However, whereas, in general, the displacement current may be neglected, Eq. (11) for the shear waves assumes the same form as in the ordinary theory of elasticity (the same applies to the problems examined [1]), and the potential of the electric field is

$$f = -\frac{\mu H}{c}\frac{\partial\psi}{\partial t}.$$

3. In order to investigate the process of propagation of magnetoelastic waves under the influence of an arbitrary system of body forces, we construct the corresponding Green's function, i. e., we assume that an impulse of magnitude Q directed, for example, along the y axis is applied to the coordinate origin at time t = 0. Applying Laplace and Fourier integral transformations to Eq. (8), putting

$$\mathbf{f}^\circ(\alpha, \beta, p) = \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(x, y, t)\exp[-pt + i(\alpha x + \beta y)]\,dt\,dx\,dy. \tag{13}$$

and assuming zero initial conditions, we find

$$\varphi^\circ = \frac{Qi\beta}{(\alpha^2+\beta^2)[(\lambda+2G)(\alpha^2+\beta^2) + p(k+\rho p)]} \qquad \left(k = \frac{\sigma\mu^2H^2}{c^2}\right). \tag{14}$$

Then the unknown potential is given by the inversion formula

$$\varphi = \frac{Q}{8\pi^3(\lambda+2C)}\int_{\gamma-i\infty}^{\gamma+i\infty} c^{pt}\,dp\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\beta\exp[-i(\alpha x+\beta y)]}{\alpha^2+\beta^2+\mu^2 p}\,\frac{d\alpha d\beta}{\alpha^2+\beta^2}, \tag{15}$$

where

$$\mu^2 = \frac{p^2}{a^2}\left(1+\frac{k}{\rho p}\right), \qquad a = \left(\frac{\lambda+2G}{\rho}\right)^{1/2}.$$

After introducing polar coordinates and using the integral representation of Bessel functions, we have

$$\varphi = \frac{Qy}{4\pi^2 ir(\lambda+2G)}\int_{\gamma+i\infty}^{\gamma+j\infty} e^{pt}dp\int_0^\infty \frac{J_1(rR)}{R^2+\mu^2}\,dR \qquad (r = \sqrt{x^2+y^2}). \tag{16}$$

The inner integral is evaluated (see, for example, [2], page 692):

$$\int_0^\infty \frac{J_1(rR)}{R^2+\mu^2}\,dR = \frac{1-\mu r K_1(\mu r)}{\mu^2 r}.$$

Then, using the formula ([3], page 284)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_1(\mu r)\,e^{pt}\frac{dp}{\mu} = \begin{cases} 0 & (t<r/a) \\ (2\rho a^2/kr)\exp(-kt/2\rho)\,\mathrm{sh}\,[(t^2-r^2/a^2)^{1/2}k/2\rho] & (t>r/a) \end{cases},$$

we find

$$\varphi = \frac{Qy}{2\pi r^2}\left[\frac{1}{k}\left(1-\exp\frac{-kt}{\rho}\right)-\frac{2}{k}\exp\left(\frac{-kt}{2\rho}\right) sh\,\frac{k}{2\rho}\left(t^2-\frac{r^2}{a^2}\right)^{1/2}\right] \qquad \left(t>\frac{r}{a}\right),$$
$$\varphi = \frac{Qy}{2\pi r^2 k}\left(1-\exp\frac{-kt}{\rho}\right) \qquad \left(t<\frac{r}{a}\right). \tag{17}$$

We reduce the expression for the shear wave potential $\psi$ to the form:

$$\psi = -\frac{Qx}{4\pi^2 i r^2 G}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,[1-\nu r K_1(\nu r)]\,\frac{dp}{\nu^2}, \qquad \nu^2 = \frac{p^2}{G/\rho}\left(1+\frac{k/\rho p}{1+4\pi\sigma/\varepsilon p}\right) \tag{18}$$

(in its real form it is represented by very complex integrals).

As $k \to 0$, when there is either no magnetic field or the medium is nonconducting, expressions (17), (18) give the potentials of the elastic waves produced by a point source.

Analysis of the results shows that with the above assumptions the elastic potentials $\varphi$ and $\psi$ also contain both wave terms (with the usual propagation velocities a and b) and components corresponding to instantaneously propagating perturbations (qualitatively analogous results were obtained in [1]).

## REFERENCES


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